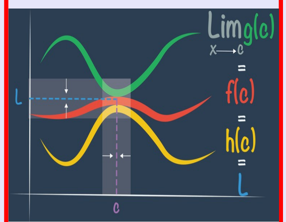
Math 261

Spring 2022

Lecture 13



Find $\frac{dy}{dx}$

$$\frac{1}{x} + \frac{1}{y} = 1$$

1) LCD xy

$$y + x = xy$$

$$\frac{dy}{dx} + 1 = 1 \cdot y + x \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} - x\frac{dy}{dx} = y - 1$$

$$\frac{dy}{dx}(1 - x) = y - 1 \Rightarrow \boxed{\frac{dy}{dx} = \frac{y-1}{1-x}}$$

2) $x^{-1} + y^{-1} = 1$

$$-1x^{-2} - 1y^{-2}\frac{dy}{dx} = 0$$

$$\frac{1}{x^2} + \frac{1}{y^2}\frac{dy}{dx} = 0$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{-1}{x^2} \qquad \frac{dy}{dx} = \frac{\frac{-1}{x^2}}{\frac{1}{y^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2}{x^2}}$$

3) $\frac{1}{x} + \frac{1}{y} = 1$

$$\frac{1}{y} = 1 - \frac{1}{x}$$

$$\frac{1}{y} = \frac{x-1}{x}$$

$$y = \frac{x}{x-1}$$

$$\frac{dy}{dx} = \frac{1\cdot(x-1) - x(1)}{(x-1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{(x-1)^2}}$$

March 23, 2022

Find $\frac{dy}{dx}$

$$\tan(x-y) = \frac{y}{1+x^2}$$

$$\sec^2(x-y)\cdot\left[1-\frac{dy}{dx}\right] = \frac{\frac{dy}{dx}(1+x^2) - y\cdot 2x}{(1+x^2)^2}$$

Cross-multiply

$$(1+x^2)^2\cdot\sec^2(x-y)\cdot\left[1-\frac{dy}{dx}\right] = (1+x^2)\frac{dy}{dx} - 2xy$$

$$(1+x^2)^2\sec^2(x-y) - (1+x^2)^2\sec^2(x-y)\cdot\frac{dy}{dx} = (1+x^2)\frac{dy}{dx} - 2xy$$

$$(1+x^2)^2\sec^2(x-y) + 2xy = (1+x^2)\cdot\frac{dy}{dx} + (1+x^2)^2\sec^2(x-y)\frac{dy}{dx}$$

$$(1+x^2)^2\sec^2(x-y) + 2xy = \left[1+x^2 + (1+x^2)^2\sec^2(x-y)\right]\frac{dy}{dx}$$

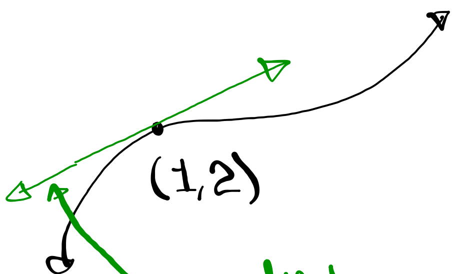
$$\boxed{\frac{dy}{dx} = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{1+x^2+(1+x^2)^2\sec^2(x-y)}}$$

Find eqn of the tan. line at (1,2) to the curve given by $x^2+2xy-y^2+x=2$

verify (1,2)

$1^2+2(1)(2)-2^2+1=2$

$1 \cancel{+4} \cancel{-4} +1 = 2$ ✓



$$m = \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{7}{2}$$

$$2x + 2\left[1\cdot y + x\cdot\frac{dy}{dx}\right] - 2y\cdot\frac{dy}{dx} + 1 = 0$$

$$2(1) + 2[2 + 1\cdot m] - 4m + 1 = 0$$

$$2 + 4 + 2m - 4m + 1 = 0$$

$$-2m = -7 \quad \boxed{m = \frac{7}{2}}$$

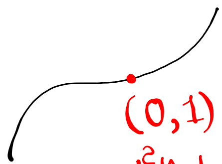
$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{7}{2}(x-1) \Rightarrow \boxed{y =\quad}$$

If $xy + y^3 = 1$, find the value of y'' at the point where $x = 0$.

$0 \cdot y + y^3 = 1$

$y^3 = 1 \to \boxed{y = 1}$



$\frac{d^2y}{dx^2}\Big|_{(0,1)}$

$xy + y^3 = 1$

$1 \cdot y + x \cdot y' + 3y^2 \cdot y' = 0$

$\to y + xy' + 3y^2 y' = 0$

$y'(x + 3y^2) = -y$

$\Rightarrow y' = \frac{-y}{x + 3y^2}$

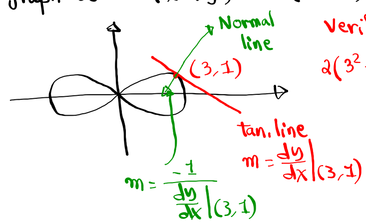
$y'' = \frac{-y'(x+3y^2) + y(1 + 6y \cdot y')}{(x+3y^2)^2}$

$y'\Big|_{(0,1)} = \frac{-1}{0 + 3(1)^2} = \boxed{\frac{-1}{3}}$

$y''\Big|_{(0,1)} = \frac{\frac{1}{3}(0+3) + 1(1 + 6 \cdot \frac{-1}{3})}{(0+3)^2}$

$= \frac{1 + -2}{9} = \boxed{\frac{-1}{9}}$

Find the equation of the normal line to the graph of $2(x^2+y^2)^2 = 25(x^2-y^2)$ at $(3,1)$



Verify (3,1):

$2(3^2+1^2)^2 = 25(3^2-1^2)$

$2 \cdot 10^2 = 25 \cdot 8$

$200 = 200$ ✓

$2(x^2+y^2)^2 = 25(x^2-y^2)$

$2 \cdot 2(x^2+y^2)^{2-1} \cdot \left[2x + 2y\frac{dy}{dx}\right] = 25\left[2x - 2y \cdot \frac{dy}{dx}\right]$

$4(3^2+1^2) \cdot [6 + 2m] = 25[6 - 2m]$

$40(6+2m) = 25(6-2m)$

$80(3+m) = 50(3-m)$ Divide by 10

$8(3+m) = 5(3-m)$

$24 + 8m = 15 - 5m$

$8m + 5m = 15 - 24$

$13m = -9$

$\boxed{m = \frac{-9}{13}}$

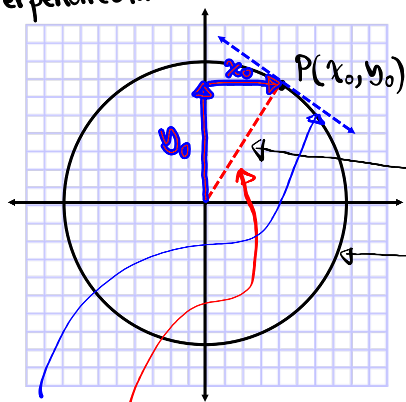
slope of tan. line at (3,1)

$m = \frac{13}{9}$

slope of the normal line

$y - 1 = \frac{13}{9}(x-3) \Rightarrow \boxed{y = \ }$

Show that any tangent line at a point P to a circle with center at the origin O(0,0) is perpendicular to the radius OP.



$m = \dfrac{y_0}{x_0}$

$x^2 + y^2 = r^2$

$2x + 2y\,y' = 0$

$y' = \dfrac{-x}{y}$

at $P(x_0, y_0)$

$m = \dfrac{-x_0}{y_0}$

$m \cdot m = \dfrac{-x_0}{y_0} \cdot \dfrac{y_0}{x_0} = -1$

Product of slopes is -1

Lines must be $\perp$.

Show that these curves $y = ax^3$ and $x^2 + 3y^2 = b$ are orthogonal.

$y = ax^3$

$\dfrac{dy}{dx} = 3ax^2$

$x^2 + 3y^2 = b$

$2x + 6y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-2x}{6y} = \dfrac{-x}{3y}$

Now we need to show that the product of these derivatives is -1.

$\cancel{3}ax^2 \cdot \dfrac{-x}{\cancel{3}y} = \dfrac{-ax^3}{y} = \dfrac{-\cancel{y}}{\cancel{y}} = \boxed{-1}$

Show $y=\frac{1}{x+C}$ and $y=a\sqrt[3]{x+K}$ are orthogonals.

$$y'=\frac{0\cdot(x+C)-1\cdot 1}{(x+C)^2}$$

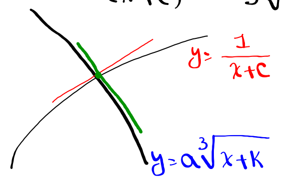
$$\boxed{y'=\frac{-1}{(x+C)^2}}$$

$$y=a(x+K)^{1/3}$$

$$y'=a\cdot\frac{1}{3}(x+K)^{\frac{1}{3}-1}\cdot 1$$

$$\boxed{y'=\frac{a}{3\sqrt[3]{(x+K)^2}}}$$

Show $\frac{-1}{(x+C)^2}\cdot\frac{a}{3\sqrt[3]{(x+K)^2}}=-1$



Intersection Point

$$y=y$$

$$\frac{1}{x+C}=a\sqrt[3]{x+K}$$

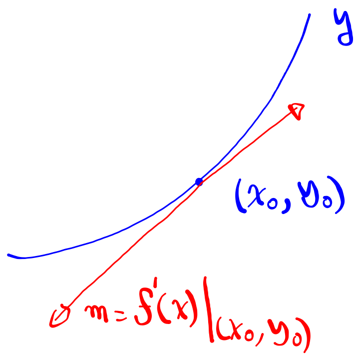
Square both Sides

$$\frac{1}{(x+C)^2}=a^2\sqrt[3]{(x+K)^2}$$

$$\frac{-1}{(x+C)^2}\cdot\frac{a}{3\sqrt[3]{(x+K)^2}}=$$

$$-a^2\cancel{\sqrt[3]{(x+K)^2}}\cdot\frac{a}{3\cancel{\sqrt[3]{(x+K)^2}}}=\frac{-a^3}{3}=-1$$ only if $a^3=3$

these two family of curves are orthogonal only if $\boxed{a=\sqrt[3]{3}}$

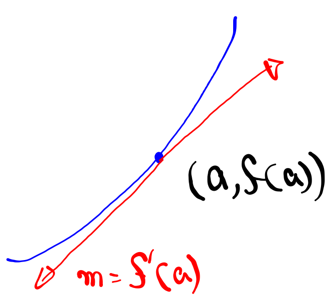


Equation of tan. line

$$y-y_0=m(x-x_0)$$

$$y=y_0+m(x-x_0)$$

$$y=f(x_0)+f'(x_0)(x-x_0)$$



$$y=f(a)+f'(a)(x-a)$$

$$f(x)\approx\underbrace{f(a)+f'(a)(x-a)}_{L(x)}$$

$L(x)$ ↑ Linear Approximation

Approximate $\sqrt{4.1} \approx \sqrt{4} = 2$

Let $f(x) = \sqrt{x}$, $a = 4$ $\quad f'(x) = \dfrac{1}{2\sqrt{x}}$

By linear appoximation

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$

$$L(x) = 2 + \frac{1}{4}(x-4)$$

$$L(4.1) = 2 + \frac{1}{4}(4.1-4)$$

$$= 2 + \frac{1}{4}\cdot(.1)$$

$$= 2 + \frac{1}{4}\cdot\frac{1}{10} = 2 + \frac{1}{40} = \frac{81}{40}$$

$$= \boxed{2.025}$$

$\boxed{2.0248}$

$\sqrt{4.1} \approx \boxed{2.025}$

Approximate $\sqrt[3]{65} \approx \sqrt[3]{64} = 4$

Let $f(x) = \sqrt[3]{x}$, $a = 64$ $\quad f(64) = \sqrt[3]{64} = 4$

$f(x) = x^{1/3}$ $\quad f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ $\quad f'(64) = \dfrac{1}{3\sqrt[3]{64^2}} = \dfrac{1}{48}$

Linear Approximation $L(x) = f(a) + f'(a)(x-a)$

$$L(x) = f(64) + f'(64)\cdot(x-64)$$

$$L(x) = 4 + \frac{1}{48}(x-64)$$

$$L(65) = 4 + \frac{1}{48}(65-64)$$

$$= 4 + \frac{1}{48}(1) = 4 + \frac{1}{48} = \frac{193}{48}$$

$$= 4.0208\overline{3}$$

From Calc.

$\boxed{4.020}7\ldots \approx \sqrt[3]{65}$ $\quad \approx 4.021$

March 23, 2022

Approximate $\tan 44^\circ \approx \tan 45^\circ = 1$

Let $f(x) = \tan x \quad a = 45^\circ \quad f(45^\circ) = \tan 45^\circ = 1$

$f'(x) = \sec^2 x \quad f'(45^\circ) = \sec^2 45^\circ = (\sqrt{2})^2 = 2$

Linear Approximation

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(45^\circ) + f'(45^\circ)(x-45^\circ)$$

$$L(x) = 1 + 2(x-45^\circ)$$

$$\tan 44^\circ \approx L(44^\circ) = 1 + 2(44^\circ - 45^\circ)$$

$180^\circ = \pi$ radians

$1^\circ = \frac{\pi}{180}$ Radians

$$= 1 + 2(-1^\circ)$$

$$= 1 - 2 \cdot 1^\circ = 1 - 2 \cdot \frac{\pi}{180}$$

$$\underbrace{\tan 44^\circ}_{.966} \approx 1 - \underbrace{\frac{\pi}{90}}_{.965}$$

$.966 \approx .965$

Use Your Calc to find $\frac{1}{4.002}$

$\boxed{.249}875 \ldots \approx .250$

Approximate $\frac{1}{4.002} \approx \frac{1}{4} = .25$

Let $f(x) = \frac{1}{x} \quad a = 4 \quad f(4) = \frac{1}{4} = .25$

$f'(x) = \frac{-1}{x^2} \quad f'(4) = \frac{-1}{4^2} = \frac{-1}{16}$

Linear Approximation $L(x) = f(a) + f'(a)(x-a)$

$$L(x) = \frac{1}{4} + \frac{-1}{16}(x-4)$$

$$f(4.002) \approx L(4.002) = \frac{1}{4} - \frac{1}{16}(4.002-4)$$

$$= \frac{1}{4} - \frac{1}{16}(.002)$$

$$= \frac{1}{4} - \frac{1}{\cancel{16}_8} \cdot \frac{\cancel{2}}{1000}$$

$$= \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000}$$

$\boxed{.249}875 \ldots = \boxed{.249}875$

$.250 = .250$

Approximate $(2.1)^5 \approx 2^5 = 32$

$f(x) = x^5 \qquad a = 2 \qquad f(2) = 32$

$f'(x) = 5x^4 \qquad f'(2) = 5(2)^4 = 80$

Linear Approximation

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(2) + f'(2)(x-2)$

$L(x) = 32 + 80(x-2)$

$(2.1)^5 = f(2.1) \approx L(2.1) = 32 + 80(2.1-2)$

$= 32 + 80(.1)$

$= 32 + 8$

$= 40$

$(2.1)^5 \approx 40$

$40.841 \approx 40$

Class QZ 9

Find eqn of tan. line at $(2,1)$ to the curve given by $x^2 + 4xy + y^2 = 13$.

Final Ans. in Slope-Int. Form.

Verify $(2,1)$:

$2^2 + 4(2)(1) + 1^2 = 13$ ✓

$2x + 4\left[1 \cdot y + x \cdot \frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$

$2(2) + 4(1 + 2m) + 2m = 0$

$4 + 4 + 8m + 2m = 0$

$10m = -8$

$m = \frac{-8}{10}$

$\boxed{m = \frac{-4}{5}}$

$y - 1 = \frac{-4}{5}(x-2)$

$y = \frac{-4}{5}x + \frac{8}{5} + 1$

$\boxed{y = \frac{-4}{5}x + \frac{13}{5}}$